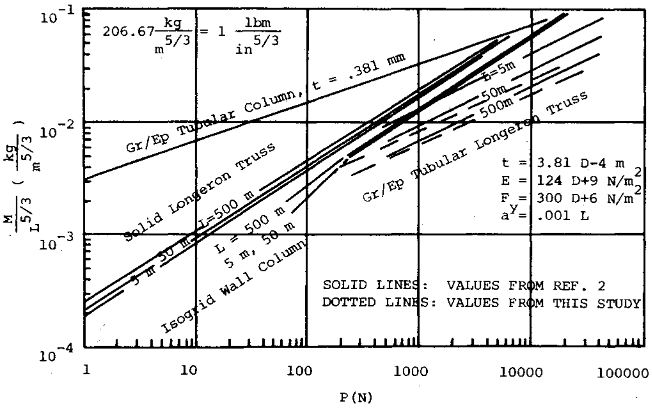

Fig. 4 Masses of several lightly loaded column concepts as a function of design loads.

Acknowledgment

The author would like to express his gratitude to NASA and ASEE for granting the Summer Faculty Fellowship, thereby making this study possible.

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Direct Solutions for Sturm-Liouville Systems with Discontinuous Coefficients

Dewey H. Hodges*

U.S. Army Research and Technology Laboratories (AVRADCOM), NASA Ames Research Center, Moffett Field, Calif.

Introduction

In a recent paper,[1] direct analytical solutions were obtained for vibrating systems with discontinuities using the method of Ritz. The methodology is based on the use of simple power series and the results can be made as accurate as desired. In this paper, the method is applied to a class of problems described by the Sturm-Liouville equations with discontinuous coefficients. Exact solutions for a class of these problems are presented in Ref. 2. The present results are compared with the exact solutions for various values of system parameters, boundary conditions, and number of terms in the power series.

Received Aug. 7, 1978; revision received March 16, 1979. This paper is declared a work of the U.S. Government and therefore is in the public domain.

Index categories: Structural Dynamics; Heat Conduction; Vibration.

*Research Scientist, Rotorcraft Dynamics Division, Aeromechanics Laboratory. Associate Fellow AIAA.

Statement of the Problem

The problem treated is an eigenvalue problem of the Sturm-Liouville type. Longitudinal vibrations of a nonuniform beam are analyzed to illustrate the method. Similar eigenvalue problems arise from torsional vibrations of nonuniform beams and heat conduction through layered composite materials. Hamilton's law of varying action[1] yields the following expression when simple harmonic motion is assumed:

$$\int_0^L \left(EA \frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{x}} \frac{\mathrm{d}\delta\bar{u}}{\mathrm{d}\bar{x}} - \omega^2 m \bar{u} \delta\bar{u} \right) \mathrm{d}\bar{x} = 0 \tag{1}$$

where EA is the longitudinal stiffness, m is the mass per unit length, ω is the frequency of oscillation, and $\bar{u}$ is the longitudinal deflection. An appropriate scaling is:

$$u = \frac{\bar{u}}{L};\ x = \frac{\bar{x}}{L};\ \lambda^2 = \frac{m_r L^2 \omega^2}{EA_r};\ \kappa = \frac{EA}{EA_r};\ c = \frac{m}{m_r} \tag{2}$$

where EA_r and m_r are arbitrary reference values of EA and m.

Equation (1) with $u' = \mathrm{d}u/\mathrm{d}x$, now becomes

$$\int_0^1 (\kappa u' \delta u' - \lambda^2 c u \delta u) \mathrm{d}x = 0 \tag{3}$$

Since Eq. (3) can be written as a minimum principle, we know that the eigenvalues will be upper bounds when the method of Ritz is applied. The Euler equations and boundary conditions, although not required for the direct solution, may be found by integrating Eq. (3) by parts

$$-\int_0^1 [(\kappa u')' + \lambda^2 c u] \delta u \mathrm{d}x + \kappa u' \delta u |_0^1 = 0 \tag{4}$$

Note that geometric boundary conditions only affect the dimensionless displacement u whereas natural boundary conditions affect the dimensionless tension force $\kappa u'$. In the direct solution, only geometric conditions need to be satisfied by the admissible functions. The following cases are considered:

$$u(0) = u(1) = 0 \qquad \text{case 1}$$

$$u(0),\ u(1)\ \text{free} \qquad \text{case 2}$$

$$u(0) = 0,\ u(1)\ \text{free} \qquad \text{case 3} \tag{5}$$

Because of Eq. (4), case 2 implies that $\kappa(0)u'(0) = \kappa(1)u'(1) = 0$, and case 3 implies that $\kappa(1)u'(1) = 0$. It is important to emphasize, however, that Eqs. (5) are the only boundary conditions that need to be satisfied by the admissible functions.

The problem to be considered also has discontinuous coefficients κ and c. Let us assume that the coefficients κ and c are constant within each of M segments, where segments are designated to begin and end at discontinuities in either κ and c. Thus, we designate

$$\kappa = \kappa_k \qquad c = c_k \tag{6}$$

in the kth segment, which has length ℓ_k.

Direct Solution

To obtain the direct solution we note that the eigenvalue problem in Eq. (3) can be formulated in terms of the M segments coupled through geometric continuity conditions. We let $dx=\ell_k d\eta_k$ and Eq. (3) becomes

$$\sum_{k=1}^{M}\left(\frac{\kappa_k}{\ell_k}\int_0^1 u_k^+\delta u_k^+ d\eta_k-\lambda^2 c_k\ell_k\int_0^1 u_k\delta u_k d\eta_k\right)=0 \tag{7}$$

where $(\ )^+=d/d\eta_k(\ )$. The geometric continuity conditions that must be imposed at segment boundaries are

$$u_{k+1}(0)=u_k(1)\qquad k=1,2,\ldots,M-1 \tag{8}$$

and geometric boundary conditions on u in Eqs. (5) at $x=0$ and $x=1$ now apply to $u_1(0)$ and $u_M(1)$, respectively. Each function u_k is analytic and may be approximated as accurately as desired by means of a power series using the method of Ritz. We let

$$u_k=\sum_{j=0}^{N}A_{kj}\eta_k^j\qquad \delta u_k=\sum_{i=0}^{N}\delta A_{ki}\eta_k^i\qquad \begin{matrix}0\le\eta_k\le 1\\ k=1,2,\ldots,M\end{matrix} \tag{9}$$

Substitution of Eq. (9) into Eq. (7) yields a set of algebraic equations

$$\sum_{k=1}^{M}\left(\frac{\kappa_k}{\ell_k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{A_{kj}\delta A_{ki}ij}{i+j-1}-\lambda^2 c_k\ell_k\sum_{i=0}^{N}\sum_{j=0}^{N}\frac{A_{kj}\delta A_{ki}}{i+j+1}\right)=0 \tag{10}$$

With auxiliary conditions based on Eqs. (5) and (8)

$$A_{k+1,0}=\sum_{j=0}^{N}A_{kj};\quad \delta A_{k+1,0}=\sum_{i=0}^{N}\delta A_{ki}\qquad \text{cases 1, 2, and 3} \tag{11a}$$

$$A_{1,0}=0;\quad \delta A_{1,0}=0\qquad \text{cases 1 and 3} \tag{11b}$$

$$\sum_{j=0}^{N}A_{Mj}=0;\quad \sum_{i=0}^{N}\delta A_{Mi}=0\qquad \text{case 1} \tag{11c}$$

Eqs. (11a) impose geometric continuity while Eqs. (11b) and (11c) are geometric boundary conditions. Note that no boundary conditions are satisfied by the power series for case 2. The auxiliary conditions, Eqs. (11), constitute matrix operations that eliminate certain of the A_{kj} and δA_{ki} quantities in Eq. (10). These straightforward operations yield a generalized matrix eigenvalue problem that may be solved numerically through the algorithm of Ref. 3.

It is not necessary to enforce boundary conditions on $\kappa u'$ at $x=0$, 1. Furthermore, it is not necessary to enforce continuity of $\kappa u'$ at the segment boundaries in the admissible functions, although it is clear that $\kappa u'$ of the exact solution is continuous at those points. These are important points for obtaining direct solutions to more complex systems for which admissible functions that satisfy all boundary and continuity conditions may be impossible to find.

The application of the Ritz method described in this section may be regarded as a means of generating finite elements with shape functions of variable order.

Numerical Results

In Ref. 2 the exact solution was obtained for a Sturm-Liouville problem with two discontinuities in the piecewise-constant coefficients. It is desirable to compare the results of the direct solution with those of the exact solution. We may set $M=3$ and define the following two parameters to facilitate this comparison:

$$\gamma\equiv\kappa_2/\kappa_1\qquad \theta\equiv c_2/c_1 \tag{12}$$

where

$$\begin{gathered}\ell_1=\ell_3=1/4;\quad \ell_2=1/2\\ \kappa_1=\kappa_3=2/(1+\gamma);\quad c_1=c_3=2/(1+\theta)\\ \kappa_2=\gamma\kappa_1;\quad c_2=\theta c_1\end{gathered} \tag{13}$$

Thus, when $\gamma=1$ there is no discontinuity in κ, and when $\theta=1$ there is no discontinuity in c.

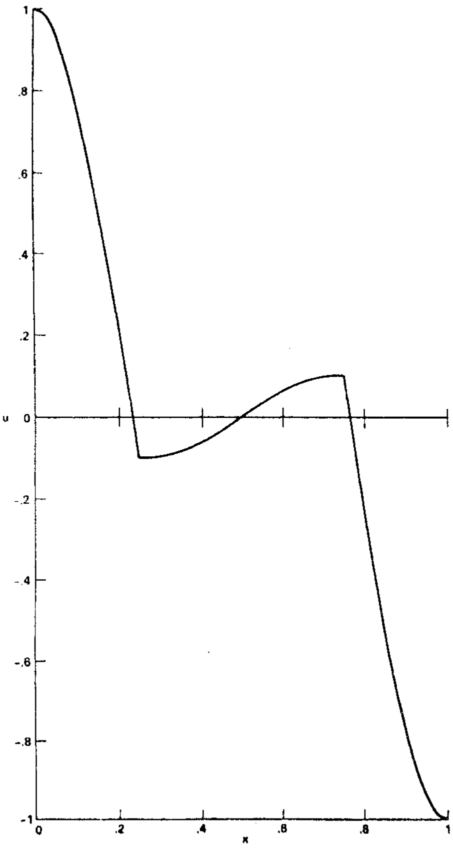

Fig. 1 Dimensionless displacement u for mode 4, case 2, $\gamma=\theta=100$; direct solution $N=8$ and exact solution.

Numerical results for the five lowest eigenvalues are presented in Table 1 for boundary conditions corresponding to cases 1 and 3. Results obtained as N varies from 1-8 are tabulated with the exact solution at the bottom. Double precision arithmetic is required to solve the eigenvalue problem on a CDC 7600 computer for N greater than about 6.† The direct solution is seen to yield good upper bounds with only a few terms and converge to the exact solution from above as more terms are added. This pattern is the same regardless of the boundary conditions and magnitude of the discontinuities for a wide range of γ and θ, results for which are not presented here due to space limitations.

For $\gamma=\theta=100$ the direct solution for $N=8$ corresponds to the exact solution expressed to eight significant figures. Such accuracy is rarely needed in engineering applications, but is demonstrated to be available through this straightforward procedure. Effects of the different boundary conditions and of the different magnitudes of discontinuities are apparent in the results, but certainly not significant.

†If Legendre polynomials are used as admissible functions, the same results are obtained and double precision arithmetic is unnecessary.

Table 1 System eigenvalues

Case 1, $\gamma=\theta=100$					
$N\backslash$Mode	1	2	3	4	5
1	0.39933479	6.9236250	12.683715	—	—
2	0.39867462	6.9188405	12.536486	12.623046	15.618240
3	0.39867461	6.2849184	12.486706	12.623046	15.605215
4	0.39867461	6.2849162	12.184127	12.569712	12.995751
5	0.39867461	6.2831861	12.184108	12.566453	12.995717
6	0.39867461	6.2831861	12.167764	12.566407	12.965206
7	0.39867461	6.2831853	12.167764	12.566371	12.965206
8	0.39867461	6.2831853	12.167696	12.566371	12.965045
Exact	0.39867461	6.2831853	12.167696	12.566371	12.965045

Case 3, $\gamma=10$, $\theta=100$					
$N\backslash$Mode	1	2	3	4	5
1	0.83316945	6.7391161	38.409861	—	—
2	0.82671094	6.7299000	14.608662	21.521724	38.526766
3	0.82670912	6.0991938	14.515832	19.259960	25.348301
4	0.82670909	6.0991374	12.030310	18.745862	25.271999
5	0.82670909	6.0973287	12.029939	17.723142	19.645471
6	0.82670909	6.0973286	11.989257	17.711322	19.599923
7	0.82670909	6.0973277	11.989254	17.581676	19.470544
8	0.82670909	6.0973277	11.989069	17.581359	19.467463
Exact	0.82670909	6.0973277	11.989068	17.579444	19.464563

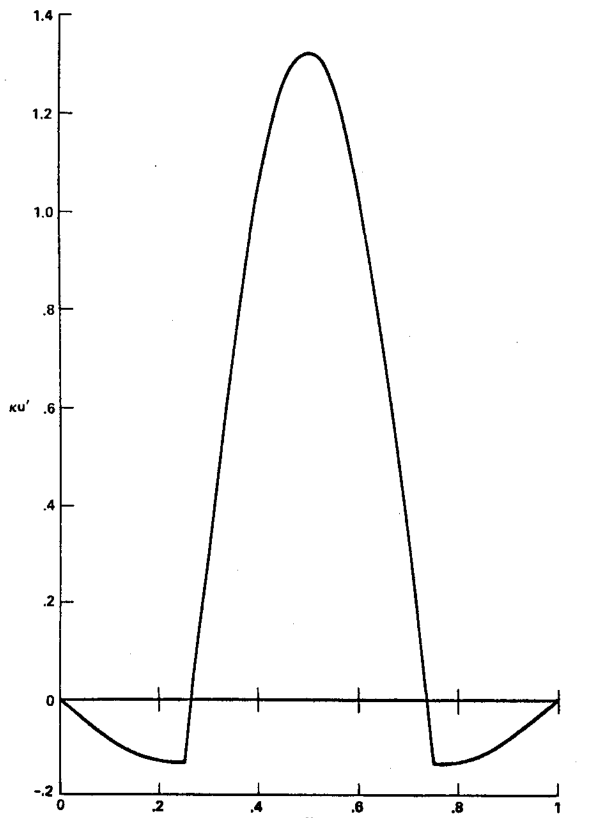

Fig. 2 Dimensionless tension force $\kappa u'$ for mode 4, case 2, $\gamma=\theta=100$; direct solution $N=8$ and exact solution.

Finally, results are presented that illustrate the accuracy of the mode-shapes obtained from the direct solution. The dimensionless displacement u and the dimensionless tension force $\kappa u'$ are both presented for mode 4 of case 2, $\gamma=\theta=100$ in Figs. 1 and 2. Results from the solution with $N=8$ are indistinguishable from the exact solution in the plots. The results for case 2 were chosen to illustrate the accuracy of the mode-shapes in order to dispel any misconceptions about the need for choosing admissible functions that satisfy natural boundary conditions. The mode-shapes in Figs. 1 and 2 were calculated using terms of a power series that do not satisfy any boundary conditions, and only continuity of u at the segment boundaries was enforced. No attempt was made to enforce natural boundary conditions to obtain $\kappa u'=0$ at $x=0,1$. The continuity of $\kappa u'$ at $x=1/4$, $3/4$ was not enforced in the admissible functions. For small values of N these conditions are usually not met, but as N is increased, the natural boundary and continuity conditions are approached rapidly. In fact, the mode-shapes can be made to satisfy the differential equation at all points along the interval to any accuracy desired by appropriate choice of N.[1] This is in contrast to conventional applications of the Rayleigh-Ritz method where the admissible functions are assumed to be analytic over the entire domain.[2] Since the exact solution is not analytic, convergence to the exact solution is precluded when analytic admissible functions are used.

Conclusion

The direct method of Ritz is applied to a class of Sturm-Liouville problems with discontinuous coefficients. Terms of a power series are used as admissible functions within segments of the domain that are chosen to have all discontinuities at their extremities. Only geometric boundary conditions are satisfied by the admissible functions, chosen to be terms of a power series, and geometric continuity is enforced at the segment boundaries. While such accuracy as seven or eight significant figures in the fifth eigenvalue is seldom needed in engineering applications, results were obtained to this precision to illustrate that this direct solution can be made as accurate as desired with relatively little effort, the only limitation being in the numerical precision of the digital computer.

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